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Classical Solution of the Nonlinear Boltzmann Equation in All R^3: Asymptotic Behavior of Solutions

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Proof is given of the existence of a classical solution to the nonlinear Boltzmann equation in all R^3. The solution, which is global in time, exists if the initial data go to zero fast enough at infinity and the mean free path is sufficiently large. The solution is smooth in the space variable if the initial value is smooth. The asymptotic behavior of solutions is also given. It is shown that as $t \to \infty$ the solution to the Boltzmann equation can be approximated by the solution to the free motion problem.

KEY WORDS: Boltzmann equation; initial value problem; kinetic theory; asymptotic behavior of solutions.

1. INTRODUCTION

In this paper we prove the existence and asymptotic behavior of a classical solution to the nonlinear Boltzmann equation in all R^3. In Section 2 we discuss existence proofs for weak solutions to the Boltzmann equation. In Section 3 we prove new results on the existence of a classical solution. The solution, which is global in time, exists if the initial data go to zero fast enough at infinity and the mean free path is sufficiently large. The last condition is realized through the smallness of the L^1 norm of a certain function h that appears in the upper bound on the initial value (see Theorems 2.1 and 2.2). The solution obtained is smooth in the space variable if the initial value is smooth.

In Section 4 we give the asymptotic behavior of the solution. We show that as $t \to \infty$ the solution to the Boltzmann equation can be approximated (in a certain norm) by the solution to the free motion problem. This

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amounts, in fact, to the existence of a corresponding wave operator associated with the Boltzmann operator. Finally, by giving an explicit bound on the collision operator as $t \to \infty$, we show that collisions become less important as compared to the translational motion of molecules of the gas. This behavior is due to escape of the gas as $t \to \infty$ from any bounded set of R^3.

Existence of a weak solution of the Boltzmann equation, Theorem 2.2, is a result essentially presented by Bellomo and Toscani in a series of papers,[4-8] utilizing the "beginning condition" of an approximation scheme of Kaniel and Shinbrot.[2] However, the Kaniel–Shinbrot method provides monotonicity, and thus the pointwise convergence of a sequence of approximate solutions, but does not provide *a priori* information about a sense of convergence within the given Banach space (except in special spaces, e.g. L^p with $1 \leqslant p < \infty$). The settings in the above-mentioned articles, and in Ref. 3 as well, all involve Banach spaces of continuous bounded functions, and thus applications of the Kaniel–Shinbrot method present certain technical difficulties. We indicate a route around these difficulties in Proposition 2.1. It may be noted that convergence of the approximate solutions is not a problem in Ref. 2, which is set in L^1.

2. BASIC EXISTENCE THEOREM

We consider the initial value problem for the Boltzmann equation

$$\frac{\partial F}{\partial t} + v \frac{\partial F}{\partial x} = J(F), \qquad t > 0, \quad F(0, v, x) = f_0(v, x) \tag{2.1}$$

where $F\colon [0, T] \times R^3 \times R^3 \to R$ is the one-particle distribution function and depends on the velocity $v \in R^3$ and the position $x \in R^3$ and whose time evolution is governed by (2.1). The collision operator J with a cutoff is given in terms of two operators

$$J(F) = Q(F, F) - FR(F)$$

where

$$\begin{aligned} Q(F, F)(v) &= \int_{S_+^2 \times R^3} B(\theta, |w - v|)\, F(v')\, F(w')\, d\omega\, dw \\ FR(F)(v) &= \int_{S_+^2 \times R^3} B(\theta, |w - v|)\, F(v)\, F(w)\, d\omega\, dw \end{aligned} \tag{2.2}$$

where $F(v)$ denotes $F(t, v, x)$.

Due to the fact that the kinetic energy and the linear momentum are conserved during the collision, we have the following relations between v, w, v', w':

$$v' = v + \omega\langle w - v, \omega\rangle, \qquad w' = w - \omega\langle w - v, \omega\rangle \tag{2.3}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in R^3, $\omega \in S^2_+ = \{\omega \in S^2 \colon \langle w - v, \omega\rangle \geqslant 0\}$, and $S^2 = \{\omega \in R^3 \colon |\omega| = 1\}$. The relations in (2.3) are equivalent to

$$v + w = v' + w', \qquad v^2 + w^2 = v'^2 + w'^2 \tag{2.4}$$

The angle $\theta \in [0, \pi/2]$ is given by $\cos\theta = \langle w - v, \omega\rangle/|w - v|$. The function $B(\theta, |v|)$ is defined on $[0, \pi/2] \times R_+/\{0\}$ and is continuous. Throughout this paper we will assume the following bound on $B(\theta, |v|)$:

$$\left|\frac{B(\theta, |v|)}{\cos\theta}\right| \leqslant c\,\frac{1 + |v|}{|v|^{\delta}}, \qquad 0 \leqslant \delta < 1 \tag{2.5}$$

where $c > 0$. For inverse power potentials, $\mathscr{F}(r) = r^{-s}$, $B(\theta, |v|) = b(\theta)\,|v|^{(s-4)/s}$. The function $b(\theta)$ is nonnegative and has a singularity at $\pi/2$ of the type $(\cos\theta)^{-\lambda}$, where $\lambda = (s+2)/s$. Assuming the usual angular cutoff hypothesis,[1] the inequality (2.5) is satisfied for all inverse power potentials with $s > 2$. The rigid-spheres model $[B(\theta, |v|) = |v| \cos\theta]$ corresponds to $\delta = 0$.

Often, due to the mathematical difficulties in dealing with problems of the type (2.1), one considers instead, after integration along characteristics, the weaker form

$$F(t) = U(t)\, f_0 + \int_0^t U(t - s)\, J(F(s))\, ds \tag{2.6}$$

where $(U(t)\, h)(v, x) = h(v, x - tv)$ for $t \in R$ and $(v, x) \in R^3 \times R^3$ is the solution to the equation

$$\frac{\partial g}{\partial t} + v\,\frac{\partial g}{\partial x} = 0, \qquad g(0, v, x) = h(v, x)$$

A solution to (2.6), considered in a function space in which $U(t)$ acts as an operator and the integral makes sense, is usually called a mild solution to (2.1). The precise definition of a mild solution will be given in Section 3.

A different form of (2.6) can be obtained by acting with $U(-t)$ on both sides of Eq. (2.6) and introducing $f(t) = U(-t)\, F(t)$. We obtain

$$f(t) = f_0 + \int_0^t U(-s)\, J(U(s)\, f(s))\, ds \tag{2.7}$$

Let us note that $J(U(s)f(s)) \neq U(s)J(f(s))$ for $s \neq 0$. In fact this property is crucial in proving existence theorems to (2.7). Except for a different notation and formulation that will be important to us later, (2.7) was first considered by Kaniel and Shinbrot[2] and later by Illner and Shinbrot,[3] Bellomo and Toscani,[4] and Toscani.[5]

In Section 3 we show how starting from a solution to (2.7), one can obtain a solution (in our case it even will be a classical solution) to the Boltzmann equation (2.1) considered in a certain Banach space.

We start the process of solving (2.7) with a few definitions. $C_b(Z)$ for $Z \subset R^n$ and $n \geq 1$ denotes the space of all real, bounded, and continuous functions defined on Z. For given positive functions h, $m \in C_b(R_+)$ let

$$M(h, m) = \{f \in C_b(R_+ \times R^3 \times R^3): \quad |f(t, v, x)| \leq c h(|x|)\, m(|v|) \text{ for some } c > 0\} \tag{2.8}$$

$M(h, m)$ with norm given by

$$\|f\| = \sup_{(t,v,x) \in R_+ \times R^3 \times R^3} (|f(t, v, x)|\, h^{-1}(|x|)\, m^{-1}(|v|)) \tag{2.9}$$

becomes a Banach space.

The next two lemmas, due to Bellomo and Toscani[4] (see also Ref. 6) are essential in proving existence theorems to (2.7).

Lemma 2.1. Let h be a continuous, nonnegative, nonincreasing function on R_+, and such that $h((1-1/2^{1/2})s) \leq h^* h(s)$ for some $0 < h^* < \infty$ and all $s \in R_+$. If $u, v \in R^3$ are orthogonal, then for all $x \in R^3$ and $t \in R_+$

$$\int_0^t h(|x + su|)\, h(|x + sv|)\, ds \leq 2h^* h(|x|) \int_0^t h(s \min\{|u|, |v|\})\, ds \tag{2.10}$$

Note that $h(s) = (1 + s^2)^{-p}$ for $p > 0$ and some $c > 0$ satisfies the assumptions on h in the above lemma.

Now we consider the integrals

$$J_1(r, \delta) = \sup_{v \in R^3} \int_{S_+^2 \times R^3} B(\theta, |w - v|) \times \frac{\exp(-rw^2)}{|w - v| \sin\theta \cos\theta}\, d\omega\, dw \qquad \text{for} \quad r > 0$$

and

$$J_2(\alpha, \delta) = \sup_{v \in R^3} \int_{S_+^2 \times R^3} \frac{B(\theta, |w - v|)(1 + v^2)^{\alpha/2}\, d\omega\, dw}{|w - v| \sin\theta \cos\theta\, (1 + v'^2)^{\alpha/2}\, (1 + w'^2)^{\alpha/2}} \qquad \text{for} \quad \alpha > 0$$

We have:

Lemma 2.2. If $B(\theta, |v|)$ satisfies the inequality (2.5), then

$$J_1(r, \delta) \leqslant c\left(\frac{2-\delta}{1-\delta}\right) r^{(3-\delta)/2} = d_{r,\delta} \tag{2.11}$$

for some $0 < c < \infty$ independent of r and δ, and

$$J_2(\alpha, \delta) \leqslant c_{\alpha,\delta} < \infty \qquad \text{for} \quad \alpha > 3 - \delta \tag{2.12}$$

Using the two last lemmas, one shows that the operator

$$(G_0 f)(t, v, x) = \int_0^t U(-s)\, J(U(s)\, f(s))\, ds)(v, x)$$

which appears in (2.7), leaves $M(h, m)$ invariant. The integral above is the Lebesgue integral computed for each $(v, x) \in R^3 \times R^3$. Let us introduce two constants

$$c_1(m, \delta) = \begin{cases} d_{r,\delta} & \text{for} \quad m(|v|) = \exp(-rv^2) \\ c_{\alpha,\delta} & \text{for} \quad m(|v|) = (1+v^2)^{-\alpha/2} \end{cases}$$

and

$$c_2(m, \delta) = \sup_{v \in R^3} \int_{S^2 \times R^3} \frac{1 + |w - v|}{|w - v|^{1+\delta}} \cos\theta\, m(|w|)\, d\omega\, dw$$

We have:

Theorem 2.1. Let $h \in L^1(R_+)$ be as in Lemma 2.1 and $m(|v|) = \exp(-rv^2)$ for $r > 0$ or $m(|v|) = (1+v^2)^{-\alpha/2}$ for $\alpha > 3 - \delta$. Then, for any Lebesgue measurable f on $R_+ \times R^3 \times R^3$ such that $f(t, v, x) \leqslant h(|x|)\, m(|v|)$ a.e. in (t, v, x), we have

$$\text{(a)} \quad (G_0^Q f)(t, v, x) = \left(\int_0^t U(-s)\, Q(U(s)\, f(s), U(s)\, f(s))\, ds\right)(v, x)$$
$$\leqslant 2h^* \|h\|_{L^1}\, c_1(m, \delta)\, h(|x|)\, m(|v|)$$

and

$$\text{(b)} \quad (G_0^R f)(t, v, x) = \left(\int_0^t f(s)\, U(-s)\, R(U(s)\, f(s))\, ds\right)(v, x)$$
$$\leqslant 2\, \|h\|_{L^1}\, c_2(m, \delta)\, h(|x|)\, m(|v|)$$

for a.e. in $(t, v, x) \in R_+ \times R^3 \times R^3$.

If, in addition, f is continuous, then

$$(G_0 f)(t, v, x) = (G_0^Q f)(t, v, x) + (G_0^R f)(t, v, x)$$

is continuous on $R_+ \times R^3 \times R^3$.

Proof. An application of Lemma 2.1 with $u = v - v'$ and $v = v - w'$ together with Lemma 2.2 gives part (a). Part (b) follows easily after integration with respect to s and noticing that

$$\int_0^t h(|a + bs|)\, ds \leqslant 2 \int_0^t h(|b|\, s)\, ds$$

for $a, b \neq 0$, $t > 0$, and nonincreasing h.

Finally, let us consider $(G_0^R f)(t, v, x)$. For $x, y, u, v \in R^3$ and $t \geqslant s \geqslant 0$ we have

$$\begin{aligned}
&(G_0^R f)(t, v, x) - G_0^R f)(s, u, y) \\
&\quad = \int_s^t \int_{S_+^2 \times R^3} f(u, y, \tau)\, f(w, y + \tau(u - w), \tau)\, B(\theta, |w - v|)\, d\omega\, dw\, d\tau \\
&\qquad + \int_0^t \int_{S_+^2 \times R^3} \{f(v, x, \tau)\, f(w, x + \tau(v - w), \tau)\, B(\theta, |w - v|) \\
&\qquad - f(u, y, \tau)\, f(w, y + \tau(u - w), \tau)\, B(\theta, |w - u|)\}\, d\omega\, dw\, d\tau \\
&\quad = I_1 + I_2
\end{aligned}$$

Since

$$\int_s^t h(|a + b\tau|)\, d\tau \leqslant 2 \int_0^{t-s} h(|b|\, \tau)\, d\tau$$

for $a, b \in R$ and $t \geqslant s \geqslant 0$, I_1 is bounded by

$$\begin{aligned}
&c \cdot 4\pi h(|y|)\, m(|u|) \int_{|w-u| \leqslant K} \left[\int_0^{(t-s)K} h(\tau)\, d\tau \right] m(|w|) \frac{1 + |w - u|}{|w - u|^{1+\delta}}\, dw \\
&\qquad + c \cdot 4\pi h(|y|)\, m(|u|) \int_{|w-u| \geqslant K} \left[\int_0^{\infty} h(\tau)\, d\tau \right] m(|w|) \frac{1 + |w - u|}{|w - u|^{1+\delta}}\, dw \\
&\quad = N_1 + N_2
\end{aligned}$$

N_2 can be made as small as we want by increasing K, and N_1 goes to zero when $|t - s| \to 0$. Note that $0 \leqslant \delta < 1$, and $h \in L^1(R_+)$. To estimate I_2 let us take $u_n \to v$ and $y_n \to x$ as $n \to \infty$. Then

$$\begin{aligned}
&P_n(\tau, w) = 2\pi f(u_n, y_n, \tau)\, f(w, y_n + \tau(u_n - w), \tau)\, B(\theta, |w - u_n|) \sin\theta \\
&\quad \xrightarrow[n \to \infty]{} 2\pi f(v, x, \tau)\, f(w, x + \tau(v - w), \tau)\, B(\theta, |w - v|) \sin\theta = P(\tau, w)
\end{aligned}$$

a.e. in $(\tau, \theta, w) \in [0, T] \times [0, \pi/2] \times R^3$. Next, for each $\varepsilon > 0$ there exists $\eta > 0$ such that if $A \subset [0, T] \times [0, \pi/2] \times R^3$ is Lebesgue measurable with vol $A < \eta$, then

$$L = \int_A |P_n(\tau, w)| \, d\tau \, d\theta \, dw < \varepsilon$$

uniformly in n. Indeed, L can be made small for any $A = [0, t] \times [0, \pi/2] \times B$ with vol B small. Furthermore, for each $\varepsilon > 0$ there exists $R > 0$ such that

$$\int_{|w| \geq R} \int_0^{\pi/2} \int_0^t |P_n(\tau, w)| \, d\tau \, d\theta \, dw < \varepsilon$$

These two conditions imply weak compactness of $\{P_n\}$ in $L^1([0, t] \times R^3)$. The weak compactness together with the pointwise convergence imply that I_2 goes to zero when $|x - y|$ and $|u - v|$ approach zero. The continuity of $(G_0^g f)(t, v, x)$ can be shown basically in the same way, utilizing Lemma 2.1.

Let us point out that the dominated convergence theorem could not be applied, since $w \to m(|w|) \, B(\theta, |w - v|)$ is not necessarily integrable.

Let $B(r) = \{f \in M(h, m) \colon \|f\| \leq r\}$ for $r > 0$. The existence theorem to (2.7) is contained in the next result, essentially due to Toscani[5] and Bellomo and Toscani.[6]

Theorem 2.2. Let h and m be as in Theorem 2.1. If

$$\lambda = 8 \|h\|_{L^1} [h^* c_1(m, \delta) + c_2(m, \delta)] < 1$$

and $f_0 \in B(1)$, then

(a) $G \colon B(2) \to B(2)$, where $Gf = f_0 + G_0 f$.

(b) $\|Gf - Gg\| \leq \lambda \|f - g\|$ for $f, g \in B(2)$.

In particular, by the contraction mapping theorem, there is a unique $f \in B(2)$ such that $Gf = f$.

We note that Theorem 2.2 holds also for $h(s) = \exp(-as^2)$, $a > 0$. Indeed, instead of Lemma 2.1, one has[6]

$$\int_0^t h(|x + us|) \, h(|x + vs|) \, ds \leq \tfrac{1}{2}(\pi/a)^{1/2} \, h(|x|)/|u + v|$$

for $u, v \in R^3$ with $\langle u, v \rangle = 0$ and $x \in R^3$, $t \in R_+$.

Further, Theorem 2.2 is also true in

$X(m, h) = \{f: f$ is Lebesgue measurable on $R_+ \times R^3 \times R^3$ and $|f(t, v, x)| \leqslant ch(|x|)\, m(|v|)$ for some $c > 0$ and a.e. in $(t, v, x)\}$

[see Theorem 2.1, parts (a) and (b)].

Next, one would like to show that if we start from $f_0 \in B(1)$ and $f_0 \geqslant 0$, then the solution to (2.7) will also stay nonnegative. This is done by using the method derived by Kaniel and Shinbrot.[2]

Let $0 < T < \infty$ be arbitrary but fixed. Starting with two elements

$$l_0, u_0 \in M_T(h, m) = \{f|_{[0,T] \times R^3 \times R^3}: f \in M(h, m)\}$$

and such that $0 \leqslant l_0(t, v, x) \leqslant v_0(t, v, x)$ for $0 \leqslant t \leqslant T$, and $(v, x) \in R^3 \times R^3$, one defines recursively two sequences $\{l_k\}$ and $\{u_k\}$ as the solutions of the equations

$$\frac{\partial l_{k+1}}{\partial t}(t, v, x) + l_{k+1}(t, v, x)[U(-t)\, R(U(t)\, u_k(t))](v, x)$$
$$= [U(-t)\, Q(U(t)\, l_k(t), U(t)\, l_k(t))](v, x) \tag{2.13a}$$

$$\frac{\partial u_{k+1}}{\partial t}(t, v, x) + u_{k+1}(t, v, x)[U(-t)\, R(U(t)\, l_k(t))](v, x)$$
$$= [U(-t)\, Q(U(t)\, u_k(t), U(t)\, u_k(t))](v, x) \tag{2.13b}$$

$$l_{k+1}(0) = u_{k+1}(0) = f_0 \qquad \text{for} \quad k \geqslant 0$$

For each $(v, x) \in R^3 \times R^3$ (2.13a) and (2.13b) are linear ordinary differential equations with unique solutions. The monotonicity properties of the operators R and Q imply that if

$$l_{k-1}(t) \leqslant l_k(t) \leqslant u_k(t) \leqslant u_{k-1}(t)$$

for $0 \leqslant t \leqslant T$, then

$$l_k(t) \leqslant l_{k+1}(t) \leqslant u_{k+1}(t) \leqslant u_k(t)$$

in the same interval. Furthermore, the same argument in Theorem 2.1 can be used to show that $l_k, u_k \in M_T(h, m)$ for $k \geqslant 1$. Thus, it follows that (2.13a), (2.13b) have unique, nonnegative solutions in $M_T(h, m)$ with $\{l_k(t)\}$ increasing and $\{u_k(t)\}$ decreasing if

$$0 = l_0(t) \leqslant l_1(t) \leqslant u_1(t) \leqslant u_0(t) \qquad \text{for} \quad 0 \leqslant t \leqslant T \tag{2.14}$$

(2.14) is called the beginning condition in Ref. 2. If one takes $u_0(t, v, x) = 2h(|x|)\, m(|v|)$, then from (2.13b) with $k=0$ and Theorem 2.1a we obtain

$$u_1(t, v, x) \leqslant u_0(t, v, x) \qquad \text{if} \quad 4h^* \|h\|_{L^1} c_1(m, \delta) \leqslant 1$$

Finally, (2.13a) with $k=0$ gives $l_1(t, v, x) \leqslant u_1(t, v, x)$ for $f_0 \in B(1)$ and $f_0 \geqslant 0$.

Proposition 2.1. Let $f_0 \in B(1)$ be nonnegative and assume that $4h^* \|h\|_{L^1} c_1(m, \delta) \leqslant 1$. Then:

(a) $l_k(t, v, x) \xrightarrow[k \to \infty]{} l(t, v, x)$ and $u_k(t, v, x) \xrightarrow[k \to \infty]{} u(t, v, x)$ pointwise for all $(t, v, x) \in [0, T] \times R^3 \times R^3$. Furthermore, l and u are Lebesgue measurable on $[0, T] \times R^3 \times R^3$ and satisfy the inequality

$$0 \leqslant l(t, v, x) \leqslant u(t, v, x) \leqslant 2h(|x|)\, m(|v|) \qquad \text{for all} \quad (t, v, x)$$

(b) If $\phi_k^1 = l_k$, $\phi_k^2 = u_k$, $\phi^1 = l$, and $\phi^2 = u$, then

$$\left[\int_0^t U(-s)\, Q(U(s)\, \phi_k^i(s), U(s)\, \phi_k^i(s))\, ds\right](v, x)$$

$$\xrightarrow[k \to \infty]{} \left[\int_0^t U(-s)\, Q(U(s)\, \phi^i(s), U(s)\, \phi^i(s))\, ds\right](v, x)$$

pointwise for all (t, v, x) and $i = 1, 2$.

(c) $$L_k^i \overset{\text{def}}{=} \left[\int_0^t \phi_{k+1}^i(s)\, U(-s)\, R(U(s)\, \phi_k^j(s))\, ds\right](v, x)$$

$$\to \left[\int_0^t \phi^i(s)\, U(-s)\, R(U(s)\, \phi^j(s))\, ds\right](v, x)$$

pointwise for all (t, v, x) and $i \neq j$, $i, j = 1, 2$.

Proof. Part (a) is clear. An application of the monotone convergence theorem gives part (b). To prove (c), let us note that for each $t \geqslant 0$,

$$L_k^i = \int_0^t \int_{S_+^2 \times R^3} \phi_{k+1}^i(s, v, x)\, \phi_k^j(s, w, x + s(v-w))\, B(\theta, |w-v|)\, d\omega\, dw\, ds$$

and

$$\phi_{k+1}^i(s, v, x)\, \phi_k^j(s, w, x + s(v-w))\, B(\theta, |w-v|)$$
$$\leqslant h(|x|)\, m(|v|)\, h(|x + s|v-w|)\, m(|w|)\, B(\theta, |w-v|)$$

where the right-hand side is integrable with respect to $d\omega\, dw\, ds$ for all $(v, x) \in R^3 \times R^3$. The dominated convergence theorem completes the proof.

Finally, as in Theorem 2.2, one can show that $u = l \in X(h, m)$ for λ small enough. This means that a nonnegative solution to (2.7) obtained in Proposition 2.1 is identical with the unique solution to (2.7) obtained in the remarks after Theorem 2.2 (h is such that $\lambda < 1$) in the space $X(m, h)$. However, if $f_0 \in M(h, m)$, Theorem 2.2 gives the unique solution in $M(h, m)$. Since $M(h, m) \subset X(m, h)$, we are done.

The method of Kaniel and Shinbrot does not provide any information about the continuity of u and l with respect to v or x. Indeed, pointwise convergence or even $L^p([0, T] \times R^3 \times R^3)$ convergence of $\{u_k\}$ and $\{l_k\}$ for some $p \geqslant 1$ does not preserve the continuity in v and x of u_k and l_k in the passage to the limit of $k \to \infty$. Thus, it is not clear that the limit of the upper and lower sequences lies in the appropriate Banach spaces in the settings of Ref. 3 [just as asserted below Eq. (2.12) of Ref. 3] and of Refs. 4–8.

We note that in these references, the authors considered the "truncated" Boltzmann operator, i.e., they defined

$$(Gf)(t, v, x) = f_0(v, x) + \left[\int_0^t U(-s)\, Q(U(s) f(s), U(s) f(s))\, ds \right] (v, x)$$

Theorem 2.2 applied to the truncated operator, together with Proposition 2.1, give the solution to (2.7) in $X(m, h)$, but not $M(h, m)$. In order to obtain a solution to (2.7) in $M(h, m)$, we have considered the full Boltzmann operator as defined in Theorem 2.2.

We want to end this section with an easy generalization of Theorem 2.2 to the case of differentiable functions with respect to x. Let h be a function given in Theorem 2.1 and $m(|v|) = (1 + v^2)^{-\alpha/2}$ for some $\alpha > 0$. For $k \geqslant 0$ an integer, we define

$$\begin{aligned} M_k(h, m) = \{ & f \in C_b(R_+ \times R^3 \times R^3) \colon (D_x^p f)(t, v, x) \in C_b(R_+ \times R^3 \times R^3) \text{ and} \\ & |D_x^p f(t, v, x)| \leqslant c_p h(|x|)\, m(|v|) \text{ for some } c_p > 0 \text{ and } |p| \leqslant k \} \end{aligned}$$

where, as usual, $p = (p_1, p_2, p_3)$, $|p| = p_1 + p_2 + p_3$ and

$$D_x^p = \left(\frac{\partial}{\partial x_1} \right)^{p_1} \left(\frac{\partial}{\partial x_2} \right)^{p_2} \left(\frac{\partial}{\partial x_3} \right)^{p_3}$$

for $x = (x_1, x_2, x_3)$. The norm in $M_k(h, m)$ is given by

$$\|f\|_k = \sup_{\substack{(t, v, x) \in R_+ \times R^3 \times R^3 \\ |p| \leqslant k}} [h(|x|)^{-1}\, m(|v|)^{-1}\, |D_x^p f(t, v, x)|]$$

and $B_k(r)$ denotes the ball of radius r in $M_k(h, m)$.

We have:

Theorem 2.3. For a given $k \geqslant 0$, $\alpha > 4 - \delta$, and $f_0 \in B_k(1)$, let us consider the operator G defined in Theorem 2.2(a). If $\|h\|_{L^1}$ is small enough, then there is a $0 < \lambda_k < 1$ such that one has:

(a) $G: B_k(2) \to B_k(2)$.

(b) $\|Gf - Gg\|_k \leqslant \lambda_k \|f - g\|_k$ for $f, g \in B_k(2)$.

Proof. It is enough to notice that $Gf = f_0 + G_0 f$, where G_0 is bilinear in f. The rest of the proof follows the lines of the proof of Theorem 2.2 [since $\alpha > 4 - \delta$, the dominated convergence theorem can be used to differentiate (2.7) under the integral with respect to v].

As before, by using Proposition 2.1, one can show that if $f_0 \in B_k(1)$ is nonnegative, then the solution f to (2.7) is also nonnegative.

3. CLASSICAL SOLUTIONS

The Banach space $M(h, m)$ used in Section 2 to obtain solutions to (2.7) is not suitable for problems concerning solutions to (2.6) or (2.1). Indeed, the group $\{U(t)\}$ does not leave $M(h, m)$ invariant. In addition, the collision term $J(f)$ displays a singular behavior in $M(h, m)$ (see Propositions 4.1 and 4.2 and the remark after Proposition 4.2). For these reasons we need a new functional setting.

We start with some notations. Let X be a Banach space. The norm in X is denoted by $|\cdot|_X$. For $\alpha \geqslant 0$, we define

$$F_\alpha(R^3, X) = \{f: f \in C(R^3, X), \lim_{|v| \to \infty} (1 + v^2)^{\alpha/2} |f(v)|_X = 0\}$$

$F_\alpha(R^3, X)$ with norm

$$\|f\|_{F_\alpha(R^3, X)} = \sup_{v \in R^3} (1 + v^2)^{\alpha/2} |f(v)|_X$$

becomes a Banach space. $F_\alpha(R^3, R)$ will be denoted by $F_\alpha(R^3)$. For $0 < T < \infty$, $C([0, T], X)$ denotes the Banach of continuous X-valued functions on $[0, T]$ with sup norm. We let $C([0, T])$ denote $C([0, T], R)$. We have

Lemma 3.1. Let X be a Banach space and $\alpha \geqslant 0$. Then $F_\alpha(R^3, C([0, T], X))$ is norm isomorphic to $C([0, T], F_\alpha(R^3, X))$.

Proof. Using well-known results on ε-tensor products (see Köthe,[9] 43.3(3′), p. 242, 44.7(2), p. 287; and Grothendieck,[10] Corollary 4, p. 128), we have

$$F_\alpha(R^3, X) \approx F_\alpha(R^3) \hat{\otimes}_\varepsilon X$$
$$C([0, T], X) \approx C([0, T]) \hat{\otimes}_\varepsilon X$$
$$X \hat{\otimes}_\varepsilon Y \approx Y \hat{\otimes}_\varepsilon X$$

where $\approx$ denotes norm isomorphism, $\hat{\otimes}_\varepsilon$ denotes the completed ε-tensor product, and Y is a Banach space. From these facts, one easily obtains that

$$\begin{aligned} F_\alpha(R^3, C([0, T], X)) &\approx F_\alpha(R^3) \hat{\otimes}_\varepsilon C([0, T]) \hat{\otimes}_\varepsilon X \\ &\approx C([0, T]) \hat{\otimes}_\varepsilon F_\alpha(R^3) \hat{\otimes}_\varepsilon X \\ &\approx C([0, T], F_\alpha(R^3, X)) \end{aligned}$$

Note that Lemma 3.1 is not true if $T = \infty$ and $C([0, T])$ is replaced by $C_b[(0, \infty)]$.

For a nonnegative integer k, let $C_b^k(R^3)$ denote the Banach space of k-times continuously differentiable bounded functions on R^3 with bounded derivatives up to order k. Let $X_{\alpha,k}$ denote

> $F_\alpha(R^3, C_b^k(R^3)) = \{f \in C_b(R^3 \times R^3)$: f is k-times continuously differentiable in x and such that $(1 + v^2)^{\alpha/2} |D_x^p f(v, x)| \xrightarrow[|v| \to \infty]{} 0$ uniformly in $x \in R^3$ and for $|p| \leqslant k\}$

with norm

$$\|f\|_{\alpha,k} = \sup_{(v,x) \in R^3 \times R^3} (1 + v^2)^{\alpha/2} |D_x^p f(v, x)|$$

Let us recall that for $t \geqslant 0$, $U(t)$: $X_{\alpha,k} \to X_{\alpha,k}$ is defined by $(U(t) f)(v, x) = f(v, x - tv)$ for $f \in X_{\alpha,k}$.

Lemma 3.2. Let $\alpha \geqslant 0$ and $k \geqslant 0$ be given. We have:

(a) $X_{\alpha+1,k+1}$ is densely and continuously embedded in $X_{\alpha,k}$.
(b) For each $t \in R$, $U(t)$ is a bounded operator in $X_{\alpha,k}$ with $\|U(t) f\|_{\alpha,k} = \|f\|_{\alpha,k}$.
(c) For each $f \in X_{\alpha,k}$, $t \to U(t) f$ is continuous from R into $X_{\alpha,k}$.
(d) $U(\cdot)$: $R \times X_{\alpha,k} \to X_{\alpha,k}$ is continuous.

Proof. Part (a) follows from the fact that $C_c^\infty(R^3 \times R^3)$ is dense in $X_{\alpha,k}$ for any $\alpha \geqslant 0$ and $k \geqslant 0$. Part (b) is obvious and part (d) follows from (b) and (c). We shall show (c) when $k=0$. For $k \geqslant 1$, the proof goes along the same lines. The group property at $\{U(t)\}$ and part (b) show that to prove (c) it is enough to consider $t \to 0$. We have, for some $R>0$,

$$\begin{aligned}\|U(t)f-f\|_{\alpha,k} &= \sup_{\substack{v \in R^3\\ x \in R^3}} (1+v^2)^{\alpha/2}\, |f(v, x-tv)-f(v,x)| \\ &\leqslant \sup_{\substack{|v| \geqslant R\\ |x| \geqslant R}} (1+v^2)^{\alpha/2}\, |f(v, x-tv)-f(v,x)| \\ &\quad + \sup_{\substack{|v| \leqslant R\\ |x| \leqslant R}} (1+v^2)^{\alpha/2}\, |f(v, x-tv)-f(v,x)| \\ &= I_1 + I_2\end{aligned}$$

Since $f \in X_{\alpha,k}$, I_1 can be made as small as we wish by increasing R. The I_2 converges to zero as $t \to \infty$ because f is uniformly continuous on $\{(|v| \leqslant R) \times (|x| \leqslant R\}$, which completes the proof.

Parts (b) and (c) of Lemma 3.2 imply that $U(t)$ is a strongly continuous group in $X_{\alpha,k}$. By the Hille–Yoshida theorem, its infinitesimal generator A has a dense domain $D(A)$ in $X_{\alpha,k}$ for each $\alpha \geqslant 0$ and $k \geqslant 0$. It is easy to check that $D(A) = X_{\alpha+1,k+1}$ and

$$(Af)(v,x) = -v\,\frac{\partial f}{\partial x}(v,x)$$

for $f \subset D(A)$.

Our next result gives the continity property of $J(f)$ in $F_\alpha(R^3, C([0,T], C_b^k(R^3)))$, which will be denoted by $Y_{\alpha,k}$.

Proposition 3.1. Let $k \geqslant 0$, $\alpha \geqslant 0$, $T>0$, and assume that the inequality (2.5) is satisfied with $0 \leqslant \delta < 1$. Then

(a) $fR(g) \in Y_{\alpha,k}$ and $\|fR(g)\|_{\alpha,k} \leqslant c\,\|f\|_{\alpha+\beta,k}\,\|g\|_{\alpha+\beta,k}$ for all $f, g \in Y_{\alpha+\beta,k}$, where $\beta > 1-\delta$, $\alpha+\beta > 4-\delta$, and a constant $c>0$ is independent of f.

(b) $\hat{Q}(f,g) \in Y_{\alpha,k}$ and $\|\hat{Q}(f,g)\|_{\alpha,k} \leqslant c\,\|f\|_{\alpha+\beta,k}\,\|g\|_{\alpha+\beta,k}$ for all $f, g \in Y_{\alpha+\beta,k}$ where $\beta > 1-\delta$, $\alpha+\beta > 3$, and the constant $c>0$ is independent of f. Here $\hat{Q}$ denotes the symmetrized version of Q in (2.2).

Proof. We will give the proof only for the case when $k=0$. The case $k \geqslant 1$ can be proven in a similar way. First, by the dominated convergence theorem, $fR(g)(t, v, x)$ is continuous in (t, v, x) for $f, g \in Y_{\alpha,0}$. For $M>0$ and $|u|, |v| \leqslant M$, $t, s \in R_+$, and $x \in R^3$, we have

$$\begin{aligned} &fR(g)(t, v, x) - fR(g)(s, u, x) \\ &\quad = [f(t, v, x) - f(s, u, x)]\, R(g)(t, v, x) \\ &\qquad + f(s, u, x)[R(g)(t, v, x) - R(g)(s, u, x)] \\ &\quad = I_1 + I_2 \end{aligned}$$

Since $f, g \in Y_{\alpha,0}$, I_1 can be made as small as we want for $|t-s|$ and $|v-u|$ small. Next, for $R>0$ we have

$$\begin{aligned} I_2 &\leqslant c\,\|f\|_{\alpha+\beta,0}\,\|g\|_{\alpha+\beta,0} \int_{\{|w| \geqslant R\}} (1+w^2)^{(\alpha+\beta+1)/2} \left(\frac{1}{|w-v|^{\delta}} + \frac{1}{|w-u|^{\delta}}\right) dw \\ &\quad + c\,\|f\|_{\alpha+\beta,0} \int_{S^3_+ \times \{|w| \leqslant R\}} [g(t, w, x)\, B(\theta, |w-v|) \\ &\quad - g(s, w, x)\, B(\theta, |w-u|)]\, d\omega\, dw \\ &= N_1 + N_2 \end{aligned}$$

Since $\alpha+\beta > 4-\delta$, $N_1 \to 0$ as $R \to \infty$. On the other hand, since $\delta < 1 < 3$ and $g \in C([0, T] \times \{|w| \leqslant M\}, C_b(R^3))$, N_2 can be made small for $|t-s|$ and $|v-w|$ small. So far we have proved that

$$\begin{aligned} fR(g) &\in C([0, T] \times \{|v|) \leqslant M\}, C_b(R^3)) \\ &\approx C(\{|v| \leqslant M\}, C([0, T], C_b(R^3)) \end{aligned}$$

for any $M>0$. Since $(1+v^2)^{\alpha/2} fR(g)(t, v, x)$ is continuous and converges to zero as $|v| \to \infty$, uniformly in $(t, x) \in [0, T] \times R^3$ for $f, g \in Y_{\alpha+\beta,0}$, where $\alpha+\beta > 4-\delta$ and $\beta > 1-\delta$, we conclude that $fR(g) \in Y_{\alpha,0}$. The inequality in part (a) can be proven in a routine way.

Next we show that if $f, g \in Y_{\alpha+\beta,0}$, $\beta > 1-\delta$ and $\alpha+\beta > 3$, then

$$\sup_{t, v, x} (1+v^2)^{\alpha/2}\, \hat{Q}(f, g)(t, v, x) < \infty$$

$$(1+v^2)^{\alpha/2}\, \hat{Q}(f, g)(t, v, x) \xrightarrow[|v| \to \infty]{} 0$$

uniformly in t and x. We have

$$\begin{aligned} &(1+v^2)^{\alpha/2}\, \hat{Q}(f, g)(t, v, x) \\ &\quad \leqslant c\,\|f\|_{\alpha+\beta,0}\,\|g\|_{\alpha+\beta,0} \\ &\qquad \times \int_{S^2_+ \times R^3} \frac{(1+v^2)^{\alpha/2}\,(1+|w-v|)\, d\omega\, dw}{|w-v|^{\delta}\,(1+v'^2)^{(\alpha+\beta)/2}\,(1+w'^2)^{(\alpha+\beta)/2}} = I(v) \end{aligned}$$

We use the estimation given in Ref. 6, p. 46:

$$\int_{S_+^2} \frac{d\omega}{(1+v'^2)^{(\alpha+\beta)/2}\,(1+w'^2)^{(\alpha+\beta)/2}} \leqslant \frac{(1+v^2)^{-1/2}\,(1+w^2)^{-1/2}}{[1+\frac{1}{2}(v^2+w^2)][1+w^2+w^2+(v\cdot w)^2]^{(\alpha+\beta-2)/2}}$$

for $\alpha+\beta>2$.

By introducing cylindrical coordinates with z axis in the $w-v$ direction, we obtain

$$I(v) \leqslant \int_{-\infty}^{\infty} dz \int_0^{\infty} d\rho \frac{(1+\rho)\,\rho^{1-\delta}(1+v^2)^{(\alpha-1)/2}\,(1+\rho^2+z^2)^{-1/2}}{(1+\rho^2+z^2+v^2)[(1+v^2)(1+z^2)+\rho^2]^{(\alpha+\beta-2)/2}}$$

Next, for $0<\varepsilon<1$, let

$$k(\delta)=\begin{cases}0, & \delta>0\\ \varepsilon, & \delta=0\end{cases}$$

We have

$$I(v) \leqslant \int_{-\infty}^{\infty} dz \int_0^{\infty} d\rho \frac{\rho^{1-\delta}(1+v^2)^{\eta}}{(1+\rho^2+z^2+v^2)^{[\alpha+k(\delta)]/2}\,(1+z^2)^{\gamma}}$$

where

$$\eta = 1+k(\delta)-\beta, \qquad \gamma=[\alpha+\beta-2-k(\delta)]/2$$

Finally

$$\int_0^{\infty} \frac{\rho^{1-\delta}\,d\rho}{(1+\rho^2+z^2+v^2)^{[2+k(\delta)]/2}} \leqslant \frac{c(\delta,\varepsilon)}{(1+z^2+v^2)^{\lambda}}$$

where $\lambda=\max\{\delta, k(\delta)\}$, and since the integral with respect to z is finite for $\alpha+\beta>3+k(\delta)$ and $\varepsilon>0$ was arbitrary, we obtain that $I(v)<\infty$ for $\alpha+\beta>3$ and $\beta\geqslant 1-\delta$ and $I(v)\xrightarrow[|v|\to\infty]{} 0$ for $\beta>1-\delta$.

By using similar arguments to those of part (a), one can show that $\hat{Q}(f,g)\in Y_{\alpha,0}$ [i.e., the continuity property of $\hat{Q}(f,g)$], which completes the proof.

We now introduce the notions of strong and mild solutions to an abstract evolution equation of the form

$$\begin{aligned} dF/dt + AF &= J(F), \qquad 0<t\leqslant T \\ F(0) &= f_0 \end{aligned} \tag{3.1}$$

where F takes values $F(t)$ in a Banach space Y, A is a linear operator in Y, and $J(F)$ is a nonlinear term.

Since in our case J does not map Y into Y, we will need an additional space $X \subset Y$ and such that $J: X \to Y$. More precisely we will need the following conditions[11]:

(i) A generates a strongly continuous semigroup $\{U(t)\}_{t \in R_+}$ in Y.

(ii) There is a Banach space X continuously and densily embedded in Y such that $U(t) X \subset X$ for $t \geqslant 0$ and $U(\cdot)|_X$ forms a strongly continuous semigroup in X.

(iii) $J: X \to Y$ is continuous.

We say that F is a strong solution to (3.1) in Y if $F \in C([0, T], X)$, $dF/dt \in C([0, T], Y)$, $F(t) \in D(A)$ for $t > 0$, where $D(A)$ is the domain of A in Y and (3.1) is satisfied for $t \in [0, T]$. We say that F is a mild solution to (3.1) in Y if $F \in C([0, T], X)$ and

$$F(t) = U(t) f_0 + \int_0^t U(t-s) J(F(s))\, ds \qquad \text{for} \quad 0 \leqslant t \leqslant T \tag{3.2}$$

where the integral on the right of (3.2) is the Riemann integral in Y.

We have:

Theorem 3.1. Let $k \geqslant 0$, $0 \leqslant \delta < 1$, and $\alpha > 4 - \delta$ be given. For any given $\varepsilon > 0$, let $m(|v|) = (1 + v^2)^{-(\alpha+\varepsilon)/2}$ and h be the function given in Theorem 2.1. If $X = X_{\alpha,k}$, $Y = X_{\alpha-1,k}$, and f is a solution to (2.7) in $M_k(h, m)$ for $f_0 \in M_k(h, m)$, then:

(a) $U(\cdot) f_T \in C([0, T], X)$, where f_T is the restriction of f to $[0, T] \times R^3 \times R^3$.

(b) $F(t) = U(t) f_T(t)$ is a mild solution to (3.1) in Y for $0 \leqslant t \leqslant T$.

Proof. Since $f_T \in C([0, T], X)$, Lemma 3.2(d) implies that $U(\cdot) f_T \in C([0, T], X)$. By Lemma 3.1 we have $F = U(\cdot) f_T(\cdot) \in Y_{\alpha,k}$. In addition, $|F(t, v, x)| \leqslant cm(|v|)$, and thus by Proposition 3.1 we obtain $J(F) \in Y_{\alpha-1,k}$. Finally, using once more Lemmas 3.1 and 3.2(d), we conclude that for $0 < t \leqslant T$, $s \to U(t-s) J(F(s))$ is continuous from $[0, T]$ into Y. This fact combined with Eq. (2.7) implies that $F(t)$ satisfies (3.2) in Y.

In order to obtain strong solutions to (3.1), we need one more condition:

(iv) A restricted to X is a bounded operator from X to Y.

Theorem 3.2. Let $k \geqslant 1$ and let $\delta, \alpha, \varepsilon, m, h$ be given as in Theorem 3.1. If $X = X_{\alpha,k}$, $Y = X_{\alpha-1,k-1}$, and f is a solution to (2.7) in

$M_k(h, m)$ for $f_0 \in M_k(h, m)$, then $F(t) = U(t) f_T(t)$ is a strong solution of (3.1) in Y for $0 \leqslant t \leqslant T$.

Proof. It is clear that the condition (iv) for A is satisfied. The rest of the proof is standard; see, for example, Martin,[12] Proposition 4.1, p. 297.

Theorem 3.2 gives a classical solution to the Boltzmann equation (2.1).

Finally, let us recall again the Boltzmann equation considered in $M_k(h, m)$:

$$f(v, x, t) = f_0(v, x) + \left[\int_0^t U(-s)\, J(U(s) f(s))\, ds \right](v, x) \tag{3.3}$$

For $\alpha > 4 - \delta$ and $f_0 \in M_k(h, m)$ one can show (see the proof of Theorem 3.1) that $s \to U(-s)\, J(U(s) f(s))\, ds$ is continuous in $X_{\alpha-1,k}$. Thus, the integral in (3.3) becomes the Riemann integral in $X_{\alpha-1,k}$ and the Boltzmann equation (3.3) in $X_{\alpha-1,k}$ can be also written in the form

$$df/dt = U(-t)\, J(U(t) f(t)), \qquad t \geqslant 0 \tag{3.4}$$

More generally, the integral in (3.3) is defined as a Lebesgue integral with respect to s, computed for each $(v, x) \in R^3 \times R^3$. The integrand, however, is not well behaved [e.g., it does not belong to $M_k(h, m)$] and equivalence of (3.3) and (3.4) fails.

4. ASYMPTOTIC BEHAVIOR OF SOLUTIONS

The asymptotic behavior of solutions to the Boltzmann equation (2.7) is based on the following two propositions:

Proposition 4.1. Let $k \geqslant 0$, $0 \leqslant \delta < 1$, and $\alpha > 4 - \delta$ ($\alpha > 3 - \delta$ for $k = 0$) be given. Take $m_\alpha(|v|) = (1 + v^2)^{-\alpha/2}$ and $1 < p,\ q < \infty$ such that $1/p + 1/q = 1$, $1 < q < 3$, and $3 - p(\alpha - 1) < p\delta < 3$. Assume h is a function given in Theorem 2.1 such that $\int_R t^2 h^q(|t|)\, dt < \infty$.

Then, for any $f \in M_k(h, m_\alpha)$, we have

$$\begin{aligned} &|[D_x^i f(s)\, R(U(s) f(s))](v, x)| \\ &\qquad \leqslant c(\alpha, p, q)\, h(|x|)\, m_{\alpha-1}(|v|)\, s^{-3/q} \end{aligned} \tag{4.1}$$

for each $|i| \leqslant k$, $s > 0$, and $(v, x) \in R^3 \times R^3$.

Proof. It is enough to consider $k=0$. Then the left-hand side of (4.1) is bounded by

$$ch(|x|)\, m_{\alpha-1}(|v|)\left[\int_{R^3} h^q(|x+(w-v)\,s|)\, dw\right]^{1/q}$$
$$\times\left[\int_{R^3} m_\alpha^p(|w|)\,\frac{(1+|w|)^p}{|w-v|^{p\delta}}\, dw\right]^{1/p}$$

A simple integration completes the proof.

We remark that when $h=c(1+x^2)^{(-1-\varepsilon)/2}$ for some $\varepsilon>0$, then we can always find q and p that satisfy the conditions of Proposition 4.1. Indeed, take $3/(1+\varepsilon)<q<3$ and $p=q/(q-1)$.

Proposition 4.2. Let $k\geqslant 0$, $0\leqslant\delta<1$, and $\alpha>4-\delta$ ($\alpha>3-\delta$ for $k=0$) be given. Take $m_\alpha(|v|)=(1+v^2)^{-\alpha/2}$ and assume h is a function given in Theorem 2.1. Then for each $1<\beta\leqslant\alpha$ there exists $1<\gamma<13/6$ such that for any $f\in M_k(h, m_\alpha)$

$$|[D_x^i\, U(-s)\, Q(U(s)\, f(s), U(s)\, f(s))](v, x)|\leqslant cm_{\alpha-\beta}(|v|)\, s^{-\gamma} \qquad (4.2)$$

for each $|i|\leqslant k$, $s>0$, and $(v, x)\in R^3\times R^3$.

The proof of Proposition 4.2 is rather lengthy and technical, and is left to the Appendix.

By sacrificing the exponent γ in (4.2) one can show that the left-hand side of (4.2) is bounded by $ch(|x|)\, m_{\alpha-\beta}(|v|)(1/s)$ for any $f\in M_k(h, m_\alpha)$, where β is any number in the interval $(3-\delta, \alpha)$. Since we are not going to use this fact, we omit its proof.

Using Propositions 4.1 and 4.2, one obtains immediately:

Theorem 4.1. Let us assume the conditions of Proposition 4.1. For any given $\varepsilon>0$, let $f\in M_k(h, m_{\alpha+\varepsilon})$ be a solution to (2.7). Then, for every $1<\beta\leqslant\alpha$ there exists $f_\infty\in X_{\alpha-\beta,k}$ such that $f(t)\xrightarrow[t\to\infty]{} f_\infty$ in $X_{\alpha-\beta,k}$. The rate of convergence is as $t^{-\lambda}$, where $0<\lambda=\min\{3/q, \gamma\}-1$ and γ is given in Proposition 4.2.

Asymptotic behavior of a strong solution is given in our next result.

Theorem 4.2. Let the assumptions of Proposition 4.1 be satisfied. In addition, assume that $k\geqslant 1$, $\alpha>4-\delta$, and $\varepsilon>0$. For $f_0\in M_k(h, m_{\alpha+\varepsilon})$ let $F(t)\in X_{\alpha,k}$ be a strong solution to (3.1) in $X_{\alpha-1,k-1}$. Then, for every $1<\beta\leqslant\alpha$ there exists $f_\infty\in X_{\alpha-\beta,k}$ such that

$$\|F(t)-U(t)\, f_\infty\|_{\alpha-\beta,k}\xrightarrow[t\to\infty]{} 0$$

The rate of convergence is the same as in Theorem 4.1.

The proof of Theorem 4.2 follows from Theorems 3.2 and 4.1. Theorem 4.2 implies that as $t \to \infty$ the solution to the Boltzmann equation can be approximated by the solution $U(t) f_\infty$ to the free motion problem. This suggests that collisions become less important as compared to the translational motion of molecules of the gas. The next corollary confirms this.

Corollary 4.1. Suppose that the assumptions of Theorem 4.2 are satisfied. If $F(t) \in X_{\alpha,k}$ is a strong solution to (3.1) in $X_{\alpha-1,k-1}$, then for each $1 < \beta \leqslant \alpha$ and $t > 0$ one has

$$\|J(F(t))\|_{\alpha-\beta,k} \leqslant c/t^\eta$$

where $1 < \eta = \min\{3/q, \gamma\}$, and q and γ are given in Propositions 4.1 and 4.2, respectively.

Finally, we also have asymptotic behavior in L^p.

Theorem 4.3. Let the assumptions of Theorem 4.2 be satisfied. If $p \geqslant 1$, Ω denotes any bounded subset of R^3 and $F(t) \in X_{\alpha,k}$ is a strong solution to (3.1) in $X_{\alpha-1,k-1}$, then $\|F(t)\|_{L^p(R^3 \times \Omega)} \xrightarrow[t \to \infty]{} 0$.

Proof. We have $F(t, v, x) \leqslant 2h(|x - tv|)\, m(|v|)$ for $t \geqslant 0$ and $(v, x) \in R^3 \times R^3$. Since $h \in L^1(R_+)$ is nonincreasing, we obtain $F(t, v, x) \xrightarrow[t \to \infty]{} 0$ a.e. in $R^3 \times R^3$. Now, $\alpha > 4 - \delta$, $0 \leqslant \delta < 1$, and boundedness of Ω imply that $F(t) \in L^p(R^3 \times \Omega)$. By the dominated convergence theorem, $F(t) \to 0$ in $L^p(R^3 \times \Omega)$ as $t \to \infty$.

Theorem 4.3 shows that the gas "escapes" as $t \to \infty$ from any bounded domain in R^3. Boundedness of Ω is important. Indeed, conservation of mass gives us the equality

$$\int_{R^3 \times R^3} F(t, v, x)\, dv\, dx = \int_{R^3 \times R^3} f_0(v, x)\, dv\, dx > 0$$

which holds for each $t > 0$.

Recently, Hamdache[13] obtained the asymptotic behavior and the decay of solutions to (2.7) in $L^p_{\alpha,r}(R^3, L^\infty_x(R^3, \exp(\beta x^2)\, dx))$, where

$$L^p_{\alpha,r}(R^3) = L^p_v(R^3, (1 + v^2)^r \exp(\alpha v^2)\, dv)$$

$\beta > 0$, $r \geqslant 0$, $\alpha \geqslant 0$, and $1 \leqslant p < \infty$. It follows from the analysis given in Ref. 13 that for $\alpha > 0$ one can take $p = \infty$, and thus we have a result analogous to Theorem 4.1 in an L^∞ setting with $h(|x|) = \exp(-\beta x^2)$ and $m(|v|) = \exp(-\alpha v^2)$. We note in addition that in Ref. 6, Theorem 3.3, Bellomo and Toscani proved a special case of Theorem 4.1 with $h(|x|) = (1 + x^2)^{-p/2}$ and $m(|v|) = \exp(-rv^2)$, where $p > 1$ and $r > 0$.

APPENDIX. PROOF OF PROPOSITION 4.2

We prove the proposition for $k=0$. The case $k \geqslant 1$ can be shown in the same way. First write

$$(U(-s)\, Q(U(s)\, f(s),\, U(s)\, f(s)))(v, x)$$
$$\leqslant c \int_{S_+^2 \times R^3} h(|x+(v-v')\, s|)\, h(|x+(v-w')\, s|$$
$$\times m_\alpha(|v'|)\, m_\alpha(|w'|)\, B(\theta, |w-v|)\, d\omega\, dw = I$$

To estimate I, we transform the integral as in Ref. 1, p. 35:

$$I = c \int_{R^3} du \int_{z \perp u} dz\, h(|x-zs|)\, h(|x-us|)$$
$$\times m(|v+z|)\, m(|v+u|)\, |u|^{-2} R(|u|, |z|)$$

Let us recall that z is integrated first over the plane perpendicular to u, and then u is integrated over the full three-dimensional space. Note $R(|u|, |z|) = \text{const} \times B(\theta, |w-v|)/\sin\theta$. As in Ref. 1, p. 38, by using (2.5), we obtain

$$R(|u|, |z|) \leqslant \text{const} \times |u|\, [z^2+u^2)^{-\delta/2} + (z^2+u^2)^{(-1-\delta)/2}]$$

Thus,

$$I \leqslant c \int_{R^3} du \int_{z \perp u} dz\, h(|x-zs|)\, h(|x-us|)\, m_\alpha(|v+z|)\, m_\alpha(|v+u|)\, |u|^{-1}$$
$$\times [(z^2+u^2)^{-\delta/2} + (z^2+u^2)^{(-1-\delta)/2}]$$

Let $1 < p_1, p_2, q_1, q_2 < \infty$ be such that $p_1^{-1} + q_1^{-1} = 1$ and $p_2^{-1} + q_2^{-1} = 1$. Writing $x = x_z + x_u$, $v = v_z + v_u$, where $x_z, v_z \perp z$ and $x_u, v_u \perp u$, we obtain

$$1 + (v+z)^2 = 1 + (v_z+z)^2 + v_u^2$$
$$1 + (v+u)^2 = 1 + (v_u+u)^2 + v_u^2$$

and

$$(x-zs)^2 = (x_z - zs)^2 + x_u^2$$
$$(x-us)^2 = (x_u - us)^2 + x_z^2$$

For $1<\beta\leqslant\alpha$ we have $I\leqslant cm_{\alpha-\beta}(|v|)\,Z_1Z_2$, where

$$Z_1=\sup_{x,v,u\in R^3}\left\{\left[\int_{z\perp u}h^{q_1}(|x_z-zs|)\,dz\right]^{1/q_1}\times\left[\int_{z\perp u}m_\beta^{p_1}(|v_z+z|)\,dz\right]^{1/p_1}\right\}$$

and

$$Z_2=\sup_{v,x\in R^3}\left\{\left[\int_{R^3}h^{q_2}(|x_u-us|)\,du\right]^{1/q_2}\times\left[\int_{R^3}m_\beta^{p_2}(|v_u+u|)\,|u|^{-p_2(1+\delta)}\,du\right]^{1/p_2}\right\}$$

The second integrals from the left in Z_1 and Z_2 are bounded by a constant independent of $v\in R^3$ if $2/\beta<p_1<2$ and $3-p_2\beta<p_2(1+\delta)<3$, respectively.

Next, for $s>0$

$$\left[\int_{z\perp u}h^{q_1}(|x_z-zs|)\,dz\right]^{1/q_1}\leqslant(2\pi)^{1/q_1}s^{-2/q_1}\left[\int_0^\infty|z|\,h^{q_1}(|z|)\,d|z|\right]^{1/q_1}$$

and

$$\left[\int_{R^3}h^{q_2}(|x_u-us|)\,du\right]^{1/q_2}=(4\pi)^{1/q_2}s^{-3/q_2}\left[\int_0^\infty|u|^2\,h^{q_2}(|u|)\,d|u|\right]^{1/q_2}$$

Since $h\in L^1_+(R)$ is nonincreasing, then $h(|x|)\leqslant\text{const}\times|x|^{-1}$ for large $|x|$. This, together with the boundedness of h, implies that the integrals on the right-hand side of the above two equalities are finite if $q_1>2$ and $q_2>3$. However, $p_1<2$ implies that $q_1>2$, and if we set $q_2=q_1+1$, we also obtain $q_2>3$.

Summarizing, we have obtained

$$I\leqslant\text{const}(p_1,\beta,\delta)\,m_{\alpha-\beta}(|v|)\,s^{-\gamma}$$

where $\gamma=2/q_1+3/(q_1+1)$. This estimation holds for $2/\beta<p_1<2$ and $3-p_2\beta<p_2(1+\delta)<3$, where $p_2=(2p_1-1)/p_1$. It is easy to check that $\gamma>1$ if $(5+7^{1/2})/6<p_1<2$. For each such p_1, $1.21<(2p_1-1)/p_1=p_2<3/2$, and since $0<\delta<1$, we also have $p_2(1+\delta)<3$. Finally, for each $\beta>1$ we can find $(5+7^{1/2})/6<p_1<2$ such that $p_2(1+\delta+\beta)>3$. This completes the proof.

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